SOLUTIONS FOR MIDTERM 1

1. The probability that a red ball is drawn is $\frac{2}{7}$ while the probability that a blue ball is drawn is $\frac{5}{7}$. There are $\binom{4}{1} = 4$ ways to draw three red balls and one blue ball in four trials; the probability of each of these is $(\frac{2}{5})^3(\frac{5}{7})$; hence

Answer: The probability that three red balls and one blue ball are drawn in four trials is $\binom{4}{1}(\frac{2}{5})^3(\frac{5}{7}) = \frac{160}{7^4}$.

2. The probability space for this problem is the infinite set $\Omega = \{H, TH, TTH, TTTH, \dots\}$. The probability of $TT\dots TH$ with n Ts is $\frac{1}{2^{n+1}}$. Let X be the random variable that counts the number of flips before H appears; the value of X on $TT\dots TH$ with n Ts is $n+1$. The expected value of X is

$$E(X) = 1 \cdot P(H) + 2 \cdot P(TH) + \cdots = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

There are several ways to compute this infinite sum. For example, multiplying by 2 one gets

$$2E(X) = 1 + \sum_{n=1}^{\infty} \frac{n+1}{2^n} = 1 + \sum_{n=1}^{\infty} (\frac{n}{2^n} + \frac{1}{2^n}) = 1 + \sum_{n=1}^{\infty} \frac{n}{2^n} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + E(X) + 1,$$

i.e. $2E(X) = 1 + E(X) + 1$. Hence,

Answer: E(X)=2.

3. (a) Answer: $-\frac{1}{3}\log_2\frac{1}{3} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{5}\log_2\frac{1}{5} - \frac{1}{6}\log_2\frac{1}{6} - \frac{1}{20}\log_2\frac{1}{20}$.

(b) Step 1: $\frac{1}{6} + \frac{1}{20} = \frac{13}{60}$. New set of probabilities in descending order: $\frac{1}{3}, \frac{1}{4}, \frac{13}{60}, \frac{1}{5}$.

Step 2: $\frac{13}{60} + \frac{1}{5} = \frac{5}{12}$. New set of probabilities in descending order: $\frac{5}{12}, \frac{1}{3}, \frac{1}{4}$.

Step 3: $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. New set of probabilities in descending order: $\frac{7}{12}, \frac{5}{12}$.

Encoding: $f(\frac{5}{12}) = 0, f(\frac{1}{5}) = 00, f(\frac{13}{60}) = 01, f(\frac{1}{20}) = 010, f(\frac{1}{6}) = 011,$
$f(\frac{7}{12}) = 1, f(\frac{1}{4}) = 10, f(\frac{1}{3}) = 11.$

Answer: $f(\frac{1}{3}) = 11, f(\frac{1}{4}) = 10, f(\frac{1}{5}) = 00, f(\frac{1}{6}) = 011, f(\frac{1}{20}) = 010.$

4. An error will be undetected if it is an error in an even number of bits, i.e. either two or four bits. The probability of a two-bit error is $\binom{4}{2}(\frac{1}{6})^2(\frac{5}{6})^2$ while the probability of a four-bit error is $(\frac{1}{6})^4$.

Answer: $\binom{4}{2}(\frac{1}{6})^2(\frac{5}{6})^2 + (\frac{1}{6})^4 = \frac{151}{6^4}$.

5. The polynomial x^4+1 is not divisible by the polynomial $g(x) = x^2+x+1$; the remainder is $x+1 \neq 0$. Therefore the CRC with generating polynomial $g(x)$ DOES detect two bit errors that are four bits apart.

6. (a) Answer: $r = -26, s = 45$, i.e. $154 \cdot (-26) + 89 \cdot 45 = 1$.

(b) Answer: $\overline{89}^{-1} = 45$.